

LOCALLY CONFORMALLY FLAT LORENTZIAN GRADIENT RICCI SOLITONS

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ABSTRACT. It is shown that locally conformally flat Lorentzian gradient Ricci solitons are locally isometric to a Robertson-Walker warped product, if the gradient of the potential function is non null, and to a plane wave, if the gradient of the potential function is null. The latter gradient Ricci solitons are necessarily steady.

1. INTRODUCTION

Let M be a differentiable manifold of dimension $n + 2$, let g be a pseudo-Riemannian metric and let f be a smooth function on M . We say that the triple (M, g, f) is a *gradient Ricci soliton* if the following equation is satisfied:

$$(1) \quad \text{Hes}_f + \rho = \lambda g,$$

where Hes denotes the Hessian, ρ denotes the Ricci tensor and λ is a real number. By contracting in the previous equation one sees that $\lambda = \frac{1}{n+2}(\Delta f + \tau)$, where τ denotes the scalar curvature and Δ denotes the Laplacian. A gradient Ricci soliton is said to be *shrinking*, *steady* or *expanding* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

Gradient Ricci solitons are a particularly interesting family of Ricci solitons. These arise as self-similar solutions of the Ricci flow $\frac{\partial}{\partial t}g(t) = -2\rho_{g(t)}$ under certain conditions. Lorentzian Ricci solitons have been investigated recently showing many essential differences with respect to the positive definite case [2, 5, 21].

A gradient Ricci soliton (M, g, f) is said to be *rigid* if (M, g) is isometric to a quotient of $N \times \mathbb{R}^k$, where N is an Einstein manifold and the potential function f is defined on the Euclidean factor as $f = \frac{\lambda}{2}\|x\|^2$ (thus generalizing the Gaussian soliton) [24]. Although rigidity is a rather restrictive condition, rigid Ricci solitons are the only solitons in many important situations as shown in [23], where it is proven that any homogeneous gradient Ricci soliton is rigid if the metric is positive definite. This result fails when passing from the Riemannian to the Lorentzian setting [5]. Indeed, indecomposable Lorentzian symmetric spaces provide examples of nontrivial steady gradient Ricci solitons in which, moreover, the gradient of the potential function ∇f is a null geodesic vector field [2].

Riemannian locally conformally flat complete shrinking and steady gradient Ricci solitons were recently classified: they are quotients of \mathbb{R}^{n+2} , \mathbb{S}^{n+2} or $\mathbb{R} \times \mathbb{S}^{n+1}$ if shrinking and the Bryant soliton if steady [10] (see also [14]). The existence of

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locally conformally flat Lorentzian steady gradient Ricci solitons of non Bryant type was proven in [2].

The purpose of this work is to investigate locally conformally flat gradient Ricci solitons in the Lorentzian setting by focusing on their local structure. The following is the main result.

Theorem 1. *Let (M, g, f) be a locally conformally flat Lorentzian gradient Ricci soliton.*

- (i) *In a neighborhood of any point where $\|\nabla f\| \neq 0$, M is locally isometric to a Robertson-Walker warped product $I \times_\psi N$ with metric $\varepsilon dt^2 + \psi^2 g_N$, where I is a real interval and (N, g_N) is a space of constant curvature c .*
- (ii) *If $\|\nabla f\| = 0$ on a non-empty open set, then (M, g) is locally isometric to a plane wave, i.e., M is locally diffeomorphic to $\mathbb{R}^2 \times \mathbb{R}^n$ with metric*

$$g = 2dudv + H(u, x_1, \dots, x_n)du^2 + \sum_{i=1}^n dx_i^2,$$

where $H(u, x_1, \dots, x_n) = a(u) \sum_{i=1}^n x_i^2 + \sum_{i=1}^n b_i(u)x_i + c(u)$ for some functions $a(u)$, $b_i(u)$, $c(u)$ and the potential function is given by $f(u, x_1, \dots, x_n) = f_0(u)$, with $f_0''(u) = -\rho_{uu} = n a(u)$.

We say that a gradient Ricci soliton is *non isotropic* if $\|\nabla f\| \neq 0$ or *isotropic* if $\|\nabla f\| = 0$. In the following sections we will study both cases separately.

Remark 2. Riemannian locally conformally flat gradient Ricci solitons are analogous to the manifolds describe in Theorem 1-(i). Due to holonomy action there exist other possibilities in Lorentzian signature, as Theorem 1-(ii) shows.

Remark 3. The character of ∇f may vary from one point to another. Thus, for example, consider the Lorentzian analog of the Gaussian soliton. Let (\mathbb{L}^{n+2}, g) be the flat Minkowski space and let $f(x_1, \dots, x_{n+2}) = \frac{\lambda}{2}(-x_1^2 + x_2^2 + \dots + x_{n+2}^2)$ be defined on \mathbb{L}^{n+2} . The gradient of f is given by $\nabla f = \lambda(x_1 + x_2 + \dots + x_{n+2})$ and the Hessian is $Hes_f = \lambda g$. Hence the soliton equation (1) is satisfied for any given λ . Note that $\|\nabla f\|^2 = \lambda^2(-x_1^2 + x_2^2 + \dots + x_{n+2}^2)$ is positive, zero or negative depending on (x_1, \dots, x_{n+2}) , so the character of ∇f varies with the point.

The paper is organized as follows. In Section 2 we recall some basic formulas and give some results showing that under certain assumptions ∇f is an eigenvector of the Ricci operator; this will be crucial in the proof of Theorem 1. Also we study two-dimensional gradient Ricci solitons and Einstein gradient Ricci solitons. We devote Section 3 to analyze locally conformally flat non isotropic gradient Ricci solitons and Section 4 to study the isotropic case, showing that the underlying structure of such a soliton is a *pp*-wave. Finally the existence of gradient Ricci solitons in *pp*-waves is discussed in general, without any further assumption, in Section 5. Thus the restriction of this discussion to locally conformally flat *pp*-waves completes the proof of Theorem 1.

2. GENERAL FORMULAE AND REMARKS

The orthogonal group decomposes the space of curvature tensors into three irreducible modules. Thus, a curvature tensor R can be written as $R = \frac{\tau}{2(n+2)(n+1)}g \odot g + \frac{1}{n}\rho_0 \odot g + W$, where ρ_0 is the traceless Ricci tensor, W is the Weyl conformal

tensor, and \odot is the Kulkarni-Nomizu product (for A, B symmetric 2-tensors, $(A \odot B)(x, y, z, w) = A(x, z)B(y, w) + A(y, w)B(x, z) - A(x, w)B(y, z) - A(y, z)B(x, w)$). Note that the curvature tensor can also be written as $R = C \odot g + W$ where $C = \frac{1}{n} \left(\rho - \frac{\tau}{2(n+1)}g \right)$ is the Schouten tensor. The summands in the previous decomposition have a geometrical meaning; thus, for example, Einstein manifolds have vanishing traceless Ricci tensor, while locally conformally flat manifolds have $W = 0$ if $n \geq 2$ and the Schouten tensor is Codazzi (i.e., its covariant derivative is totally symmetric) if $n \geq 1$. In this Section we begin the study of gradient Ricci solitons in these two particular cases.

Let (M, g, f) be a Lorentzian gradient Ricci soliton. Although the following is well-known (see, for example, [24]), we recall the formulae and sketch the proof in order to make the paper as self-contained as possible. Let Ric denote the Ricci operator defined by $g(\text{Ric}(X), Y) = \rho(X, Y)$ for any vector fields X and Y .

Lemma 4. *A Lorentzian gradient Ricci soliton with potential function f satisfies*

$$\begin{aligned} (2) \quad & \nabla \tau = 2 \text{Ric}(\nabla f), \\ (3) \quad & \tau + \|\nabla f\|^2 - 2\lambda f = \text{const}. \end{aligned}$$

Proof. Tracing the soliton equation (1) gives $\Delta f + \tau = n\lambda$, hence $\nabla \tau = -\nabla \Delta f$. The contracted second Bianchi identity ($\nabla_Z \tau = 2\text{div} \rho(Z)$) together with the Bochner formula $\text{div}(\nabla \nabla f) = \rho(\nabla f) + \nabla \Delta f$ now gives (2).

Writing the soliton equation as $\text{Ric} + \nabla \nabla f = \lambda \text{Id}$ and using (2) one has

$$\nabla \tau = 2 \text{Ric}(\nabla f) = 2\lambda \nabla f - 2\nabla_{\nabla f} \nabla f = 2\lambda \nabla f - \nabla \|\nabla f\|^2.$$

Hence $\nabla(\tau + \|\nabla f\|^2 - 2\lambda f) = 0$, which proves (3). \square

Remark 5. As a consequence of Lemma 4, there are several particular situations in which ∇f is an eigenvector of the Ricci operator. Thus, if τ is constant, from (2) it follows that ∇f is an eigenvector for the Ricci operator associated to the eigenvalue zero. Also, if ∇f is null, then from (3) one has $\tau = \text{const} + 2\lambda f$; now substitute in (2) to see that $\text{Ric}(\nabla f) = \lambda \nabla f$.

2.1. Two-dimensional steady gradient Ricci solitons. Let (M, g, f) be a two-dimensional gradient Ricci soliton. Consider the canonical para-Kähler structure \mathfrak{J} (i.e., $\mathfrak{J}^2 = \text{Id}$, $g(\mathfrak{J} \cdot, \mathfrak{J} \cdot) = -g(\cdot, \cdot)$, $\nabla \mathfrak{J} = 0$) on (M, g) . Then, proceeding as in [11, §1.3], it follows that $\mathfrak{J} \nabla f$ is a Killing vector field. Now, if ∇f is a nonnull vector field, then (M, g) is locally a warped product. On the other hand, if ∇f is a null vector field, then consider coordinates (x_1, x_2) so that the null Killing vector field is $\mathfrak{J} \nabla f = \frac{\partial}{\partial x_2}$. The metric tensor takes the form $g = a(x_1, x_2)dx_1^2 + b(x_1, x_2)dx_1 dx_2$ for some functions a, b . The fact that $\frac{\partial}{\partial x_2}$ is Killing implies that $\frac{\partial}{\partial x_2} a = 0$ and $\frac{\partial}{\partial x_2} b = 0$, therefore g is flat and $\mathfrak{J} \nabla f$ is indeed parallel.

Next we are going to extend the Hamilton cigar soliton to the Lorentzian setting. Let (M, g, f) be a two-dimensional steady gradient Ricci soliton with ∇f a timelike vector field (the spacelike case is similar). Set $M = I \times N$ with metric $g = -dt^2 + \omega(t)^2 ds^2$ and assume f only depends on t . Then a straightforward calculation from (1) gives that (M, g, f) is a steady gradient Ricci soliton if and only if

$$f''(t) - \frac{\omega''(t)}{\omega(t)} = 0 \quad \text{and} \quad -f'(t)\omega'(t) + \omega''(t) = 0.$$

Hence $f''w - f'w' = 0$, so we integrate to see that $f'(t) = \kappa w(t)$ for a constant κ . Equations above reduce to $\kappa\omega\omega' - \omega'' = 0$. Hence the possible solutions depending on the sign of κ are given by

- (i) If $\kappa = 0$; then $\omega(t) = at + b$ for constants a and b . In this case M is flat and f is constant.
- (ii) If $\kappa = r^2$; then $\omega(t) = \frac{a\sqrt{2}}{r} \tan r\sqrt{2}(at + b)$ where a and b are constants. The potential function is $f(t) = d - 2 \log \cos(\frac{r(at+b)}{\sqrt{2}})$ for a constant d , and the scalar curvature is $\tau = 2a r^2 \sec^2(r \frac{(at+b)}{\sqrt{2}})$.
- (iii) If $\kappa = -r^2$; then $\omega(t) = \frac{a\sqrt{2}}{r} \tanh r\sqrt{2}(at + b)$ for a and b constants. The potential function is $f(t) = d + 2 \log \cosh(\frac{r(at+b)}{\sqrt{2}})$ for a constant d , and the scalar curvature is $\tau = -2a r^2 \operatorname{sech}^2(r \frac{(at+b)}{\sqrt{2}})$.

Analyzing geodesic completeness in the Lorentzian case is a subtle task. Indeed Lorentzian warped products of geodesically complete manifolds need not be complete, as occurs in positive definite signature. Necessary and sufficient conditions for geodesic completeness of Lorentzian warped products were investigated in [9]. Let $M = I \times_{\omega} N$ be a warped product where $I = (\alpha, \beta)$ is a real interval and (N, g_N) is a geodesically complete manifold. Then M is timelike, spacelike and null geodesically complete if and only if for some $\gamma \in (\alpha, \beta)$ it follows that

$$\int_{\alpha}^{\gamma} \frac{\omega}{\sqrt{1+\omega^2}} dt = \int_{\gamma}^{\beta} \frac{\omega}{\sqrt{1+\omega^2}} dt = +\infty.$$

As a consequence, Lorentzian warped products given by (ii) above are not geodesically complete, while those given by (iii) are. Thus, (iii) generalizes Hamilton's cigar (see [17]) to the Lorentzian setting.

2.2. Lorentzian Einstein gradient Ricci solitons. Ricci solitons are generalizations of Einstein metrics. If (M, g) is a complete Riemannian Einstein manifold, then (M, g, f) is a gradient Ricci soliton if and only if it has $\operatorname{Hes}_f = 0$ or is a Gaussian [24]. The next result describes the local structure of Einstein gradient Ricci solitons in the Lorentzian setting.

Theorem 6. *Let (M, g) be a Lorentzian Einstein manifold. If (M, g, f) is a gradient Ricci soliton with nonconstant f , then (M, g) is Ricci flat. Moreover:*

- (i) *If $\|\nabla f\| \neq 0$, then (M, g) is locally a warped product of the form $I \times_{f'} N$ and the potential function $f(t) = \frac{\lambda}{2}t^2 + at + b$.*
- (ii) *If $\|\nabla f\| = 0$, then there exist coordinates (u, v, x_1, \dots, x_n) in which the metric has the form $g = 2dudv + \tilde{g}$, where the n -dimensional metric \tilde{g} does not depend on v . Moreover, the potential function f is given by any function $f(u)$ with $f''(u) = 0$ and the soliton is steady.*

Proof. Let (M, g, f) be an Einstein gradient Ricci soliton. Then it follows from (2) that either the potential function f is constant or otherwise (M, g) is Ricci flat. Assume (M, g) is Ricci flat. The soliton equation (1) reduces to $\operatorname{Hes}_f = \lambda g = \frac{\Delta f}{n+2}g$. This equation was previously investigated by Brinkmann [4] (see [18] for a modern exposition) showing that in a neighborhood of any point where $\|\nabla f\| \neq 0$ the manifold (M, g) decomposes locally as a warped product of a real interval $I \subset \mathbb{R}$ and an Einstein manifold (N, g_N) so that $g = \varepsilon dt^2 + (f')^2 g_N$, where f is a real function defined on I with $f' \neq 0$. Now, since (M, g) is Ricci flat, a direct computation

of the Ricci tensor for the metric $\varepsilon dt^2 + (f')^2 g_N$ shows that g_N is Einstein and f must satisfy $f''' = 0$ and $f'f''' + n\varepsilon(f'')^2 = \frac{\tau_N}{n+1}$. Hence $f(t) = \frac{\lambda}{2}t^2 + at + b$ and $\tau_N = n(n+1)\varepsilon\lambda^2$.

Now assume $\|\nabla f\| = 0$ identically. Then (3) shows that either f is constant or the gradient Ricci soliton is steady. If $\lambda = 0$ the Ricci soliton equation (1) reduces to $\text{Hes}_f = 0$. Then ∇f is a parallel isotropic vector field and the metric tensor can be written in suitable Rosen coordinates (u, v, x_1, \dots, x_n) as $g = dudv + \tilde{g}$, where the n -dimensional metric $\tilde{g}(u)$ is Ricci flat for any fixed u and does not depend on v [4, 18]. Moreover, in this coordinates $\nabla f = \frac{\partial}{\partial v}$ and the potential function depends only on the variable u . Now, the result follows by computing the Hessian of f . \square

The (not complete) gradient Ricci solitons described in Theorem 6-(i) do have a Riemannian analog. However, those given in Theorem 6-(ii) are steady gradient Ricci solitons (M, g, f) , with $\text{Hes}_f = 0$, without Riemannian counterpart.

2.3. General remarks on locally conformally flat gradient Ricci solitons.

Although locally conformally flat gradient Ricci solitons will be more deeply analyzed in Sections 3 and 4, we begin here by establishing a technical lemma. Proceeding in a similar way to that developed in [14], one has the following:

Lemma 7. *Let (M, g, f) be a locally conformally flat gradient Ricci soliton. Then ∇f is an eigenvector of the Ricci operator.*

Proof. Since (M, g) is locally conformally flat the Schouten tensor is Codazzi, this is, $(\nabla_X C)(Y, Z) = (\nabla_Y C)(X, Z)$ for all vector fields X, Y, Z . Hence

$$(4) \quad (\nabla_X \rho)(Y, Z) - \frac{X(\tau)}{2(n+1)}g(Y, Z) = (\nabla_Y \rho)(X, Z) - \frac{Y(\tau)}{2(n+1)}g(X, Z).$$

From (1) and using that $\text{Hes}_f(X, Y) = g(\nabla_X \nabla f, Y)$ one has

$$\begin{aligned} (\nabla_X \rho)(Y, Z) &= -(\nabla_X \text{Hes}_f)(Y, Z) \\ &= -Xg(\nabla_Y \nabla f, Z) + g(\nabla_{\nabla_X Y} \nabla f, Z) + g(\nabla_Y \nabla f, \nabla_X Z) \\ &= -g(\nabla_X \nabla_Y \nabla f, Z) + g(\nabla_{\nabla_X Y} \nabla f, Z). \end{aligned}$$

Substituting this expression in (4) we get

$$\begin{aligned} g(\nabla_X \nabla_Y \nabla f, Z) - g(\nabla_{\nabla_X Y} \nabla f, Z) + \frac{X(\tau)}{2(n+1)}g(Y, Z) \\ = g(\nabla_Y \nabla_X \nabla f, Z) - g(\nabla_{\nabla_Y X} \nabla f, Z) + \frac{Y(\tau)}{2(n+1)}g(X, Z). \end{aligned}$$

Thus

$$g(\nabla_X \nabla_Y \nabla f - \nabla_Y \nabla_X \nabla f - \nabla_{[X, Y]} \nabla f, Z) = -\frac{X(\tau)}{2(n+1)}g(Y, Z) + \frac{Y(\tau)}{2(n+1)}g(X, Z),$$

that is,

$$R(X, Y, Z, \nabla f) = -\frac{X(\tau)}{2(n+1)}g(Y, Z) + \frac{Y(\tau)}{2(n+1)}g(X, Z),$$

or equivalently, using (2),

$$(5) \quad R(X, Y, Z, \nabla f) = -\frac{1}{n+1}\rho(X, \nabla f)g(Y, Z) + \frac{1}{n+1}\rho(Y, \nabla f)g(X, Z).$$

Let $Z = \nabla f$ in (5) to obtain

$$\rho(Y, \nabla f)g(X, \nabla f) = \rho(X, \nabla f)g(Y, \nabla f).$$

Now choose X so that $g(X, \nabla f) = 1$ to see that for all $Y \perp \nabla f$ one has

$$0 = \rho(Y, \nabla f) = -\text{Hes}_f(Y, \nabla f)$$

and conclude that ∇f is an eigenvector of the Ricci operator. Note that ∇f is also an eigenvector of the Hessian operator $\text{hes}_f(X) = \nabla_X \nabla f$. \square

3. NON ISOTROPIC GRADIENT RICCI SOLITONS

Next we show that in a neighborhood of any point where $\|\nabla f\| \neq 0$ the underlying manifold has the local structure of a warped product, thus proving Theorem 1 - (i).

Lemma 8. *Let (M, g, f) be a locally conformally flat Lorentzian gradient Ricci soliton with $\|\nabla f\|_P \neq 0$ for some point $P \in M$. Then, on a neighborhood of P , (M, g) is a warped product of a real interval and a space of constant sectional curvature c .*

Proof. If the Weyl tensor of (M, g) vanishes, then the curvature tensor expresses as

$$\begin{aligned} R(X, Y, Z, T) &= \frac{\tau}{n(n+1)} \{g(X, T)g(Y, Z) - g(X, Z)g(Y, T)\} \\ (6) \quad &+ \frac{1}{n} \{\rho(X, Z)g(Y, T) + \rho(Y, T)g(X, Z) \\ &- \rho(X, T)g(Y, Z) - \rho(Y, Z)g(X, T)\}. \end{aligned}$$

Consider the unit vector $V = \frac{\nabla f}{\|\nabla f\|}$, which can be timelike or spacelike (we set $g(V, V) = \varepsilon$), on the tangent space $T_P M$. Complete it to a local orthonormal frame $\{V, E_1, \dots, E_{n+1}\}$ with $g(E_i, E_i) = \varepsilon_i$. Then from (5) one has

$$R(V, E_i, E_i, V) = -\frac{1}{n+1} \rho(V, V) \varepsilon_i,$$

while from (6) one gets

$$R(V, E_i, E_i, V) = \frac{\tau}{n(n+1)} \varepsilon \varepsilon_i - \frac{1}{n} \rho(V, V) \varepsilon_i - \frac{1}{n} \rho(E_i, E_i) \varepsilon.$$

Hence for all $i = 1, \dots, n+1$:

$$-\frac{1}{n+1} \rho(V, V) \varepsilon_i = -\frac{1}{n} \rho(V, V) \varepsilon_i - \frac{1}{n} \rho(E_i, E_i) \varepsilon + \frac{\tau}{n(n+1)} \varepsilon \varepsilon_i,$$

from where $\rho(E_i, E_i) \varepsilon = \frac{1}{n+1} (\tau \varepsilon - \rho(V, V)) \varepsilon_i$. Using (1) we have

$$\text{Hes}_f(E_i, E_i) = \lambda \varepsilon_i + \frac{1}{n+1} (\rho(V, V) \varepsilon - \tau) \varepsilon_i,$$

which shows that the level sets of f are totally umbilical hypersurfaces. Hence (M, g) decomposes locally as a twisted product of the form $I \times_\omega N$ (see [25, Thm. 1]). Now, since ∇f is an eigenvector of the Ricci operator by Lemma 7, it follows that $\rho(V, E_i) = 0$ for all $i = 1, \dots, n+1$, and therefore the twisted product reduces to a warped product [15, Thm. 1]. Hence (M, g) is locally a warped product $(I \times N, \varepsilon dt^2 + \psi(t)^2 g_N)$ where (N, g_N) is a Riemannian or a Lorentzian manifold of constant sectional curvature c [6]. \square

Remark 9. The potential function f in Lemma 8 is a radial function $f(t)$, and hence a direct computation from the soliton equation (1) shows that it is given as a solution to the equations:

$$f'' = \varepsilon\lambda + (n+1)\frac{\psi}{\psi''}, \quad \varepsilon\psi\psi'f' = \lambda\psi^2 - nc + \varepsilon(\psi\psi'' + n(\psi')^2).$$

Note that this equations impose restrictions on the warping function ψ , thus the warped product is not arbitrary.

4. LOCALLY CONFORMALLY FLAT ISOTROPIC GRADIENT RICCI SOLITONS

In this section we consider the case of gradient Ricci solitons with $\|\nabla f\| = 0$.

Recall that for a Riemannian metric the holonomy group acts completely reducibly, i.e., the tangent space decomposes into subspaces on which it acts trivially or irreducibly, but for indefinite metrics the situation is more subtle. Indecomposable but not irreducible Lorentzian manifolds admit a parallel degenerate line field \mathcal{D} , and thus the curvature satisfies (see, for example, [12])

$$(7) \quad R(\mathcal{D}, \mathcal{D}^\perp, \cdot, \cdot) = 0, \quad R(\mathcal{D}, \mathcal{D}, \cdot, \cdot) = 0, \quad \text{and} \quad R(\mathcal{D}^\perp, \mathcal{D}^\perp, \mathcal{D}, \cdot) = 0.$$

If \mathcal{D} is spanned by a parallel null vector field, then (M, g) is said to be a *pp-wave* if

$$(8) \quad R(\mathcal{D}^\perp, \mathcal{D}^\perp, \cdot, \cdot) = 0.$$

(M, g) is called a *pr-wave* if (8) is satisfied but \mathcal{D} is not necessarily spanned by a parallel vector field. Clearly any *pp-wave* is a *pr-wave*, and the converse is true if the Ricci tensor is isotropic (i.e., the image of the Ricci operator is totally isotropic) [19]. The general form of an $(n+2)$ -dimensional *pp-wave* is the following: the ambient space is \mathbb{R}^{n+2} with coordinates (u, v, x_1, \dots, x_n) , $n \geq 1$, and the Lorentzian metric is given by

$$(9) \quad g_{ppw} = 2dudv + H(u, x_1, \dots, x_n)du^2 + \sum_{i=1}^n dx_i^2,$$

where $H(u, x_1, \dots, x_n)$ is an arbitrary smooth function.

Lemma 10. *Any isotropic locally conformally flat Lorentzian gradient Ricci soliton is steady and the underlying manifold is locally a pp-wave.*

Proof. Let (M, g, f) be a gradient Ricci soliton with $\|\nabla f\| = 0$. In what follows we will show that ∇f spans a parallel null line field and furthermore that (8) holds. Set $V = \nabla f$. Since V is a null vector, there exist orthogonal vectors S, T satisfying $g(S, S) = -g(T, T) = \frac{1}{2}$ such that $V = S + T$. Define $U = S - T$, which is a null vector such that $g(U, V) = g(S, S) - g(T, T) = 1$, and consider a pseudo-orthonormal basis $\{U, V, E_1, \dots, E_n\}$. For any vector field Z , from equations (5) and (6) we get

$$\begin{aligned} (10) \quad R(Z, E_i, E_j, V) &= -\frac{1}{n+1}\rho(Z, V)\delta_{ij} + \frac{1}{n+1}\rho(E_i, V)g(Z, E_j) \\ &= \frac{\tau}{n(n+1)}g(Z, V)\delta_{ij} - \frac{\tau}{n(n+1)}g(E_i, V)g(Z, E_j) \\ &\quad - \frac{1}{n}\rho(Z, V)\delta_{ij} - \frac{1}{n}\rho(E_i, E_j)g(Z, V) \\ &\quad + \frac{1}{n}\rho(Z, E_j)g(E_i, V) + \frac{1}{n}\rho(E_i, V)g(Z, E_j). \end{aligned}$$

We use the fact that $\text{Ric}(V) = \lambda V$ (Remark 5) to see that

$$\rho(V, V) = 0, \rho(U, V) = \lambda, \rho(V, E_i) = 0 \text{ for all } i = 1, \dots, n.$$

On the other hand compute $R(U, E_i, E_j, V)$ in expression (10) to get that

$$\begin{aligned} R(U, E_i, E_j, V) &= -\frac{1}{n+1}\lambda\delta_{ij} \\ &= \frac{\tau}{n(n+1)}\delta_{ij} - \frac{1}{n}\lambda\delta_{ij} - \frac{1}{n}\rho(E_i, E_j). \end{aligned}$$

Hence $\rho(E_i, E_j) = 0$ if $i \neq j$ and $\rho(E_i, E_i) = \frac{\tau-\lambda}{n+1}$ for all $i = 1, \dots, n$. Now, compute

$$\tau = 2\rho(U, V) + n\rho(E_i, E_i) = \frac{(n+2)\lambda + n\tau}{n+1}$$

to see that $\tau = (n+2)\lambda$. Hence the scalar curvature τ is constant and from (2) we have $0 = \nabla\tau = 2\text{Ric}(V) = 2\lambda V$. Therefore we conclude that $\lambda = 0 = \tau$ and the only possibly nonzero Ricci component is $\rho(U, U)$, so (M, g, f) is a steady gradient Ricci soliton with nilpotent Ricci operator.

Since the soliton is steady, from (1) we have $\text{hes}_f = -\text{Ric}$. Now since $\text{Ric}(V) = 0$, it follows that $\nabla_V V = 0$, which shows that V is a geodesic vector field.

The gradient of the potential function is a recurrent vector field (i.e., the null line field $\mathcal{D} = \text{span}\{\nabla f\}$ is parallel) if and only if $\nabla_X \nabla f = \text{hes}_f(X) = \sigma(X)\nabla f$ for some 1-form σ and for all X . Since (M, g, f) is a steady gradient Ricci soliton, it follows from the expressions above for the Ricci operator that

$$\begin{aligned} \text{hes}_f(U) &= -\text{Ric}(U) = -\rho(U, U)V, \\ \text{hes}_f(V) &= -\text{Ric}(V) = 0, \\ \text{hes}_f(E_i) &= -\text{Ric}(E_i) = 0, \end{aligned}$$

showing that V is a recurrent vector field with 1-form σ given by $\sigma(U) = -\rho(U, U)$, $\sigma(V) = 0$ and $\sigma(E_i) = 0$ for all $i = 1, \dots, n$.

It follows now from (6), the expressions of the Ricci tensor above and the vanishing of the scalar curvature that

$$R(\mathcal{D}^\perp, \mathcal{D}^\perp, \cdot, \cdot) = 0.$$

This shows that (M, g) is a *pr*-wave. Moreover note that the Ricci tensor is isotropic and thus that (M, g) is indeed a *pp*-wave [19]. \square

Remark 11. Note that although (M, g) is a *pp*-wave, and hence it admits a null parallel vector field, ∇f is not in general parallel.

5. RICCI SOLITONS ON *pp*-WAVES

In this section we analyze the existence of gradient Ricci solitons on *pp*-waves. Theorem 1-(ii) will follow as a consequence of Lemma 10 and the analysis performed here. *pp*-waves are the underlying structure corresponding to many Lorentzian geometric properties without Riemannian analog, thus they are a natural family to look for new examples of complete gradient Ricci solitons.

Henceforth we set $M = \mathbb{R}^{n+2}$ with coordinates (u, v, x_1, \dots, x_n) , and metric g_{ppw} given by (9) for some arbitrary function $H(u, x_1, \dots, x_n)$. The possibly nonzero

components of the Levi-Civita connection in the basis of coordinate vector fields $\{\partial_u = \frac{\partial}{\partial_u}, \partial_v = \frac{\partial}{\partial_v}, \partial_i = \frac{\partial}{\partial_{x_i}}\}$ are

$$(11) \quad \nabla_{\partial_u} \partial_u = \frac{1}{2} \partial_u H \partial_v - \frac{1}{2} \sum_{i=1}^n \partial_i H \partial_i, \quad \nabla_{\partial_u} \partial_i = \frac{1}{2} \partial_i H \partial_v, \quad i = 1, \dots, n.$$

This shows that the null vector field ∂_v is parallel. Moreover, the possibly nonvanishing components of the curvature tensor are given (up to the usual symmetries) by

$$(12) \quad R_{uiuj} = -\frac{1}{2} \partial_{ij}^2 H, \quad i, j = 1, \dots, n.$$

The scalar curvature τ is zero, since the Ricci tensor is determined by

$$(13) \quad \rho_{uu} = -\frac{1}{2} \sum_{i=1}^n \partial_{ii}^2 H.$$

Therefore, a *pp*-wave is Einstein (and hence Ricci flat) if and only if the space-Laplacian of the defining function H vanishes identically.

Theorem 12. *(M, g_{ppw}, f) is a nontrivial gradient Ricci soliton if and only if it is steady and the potential function f satisfies $f(u, x_1, \dots, x_n) = f_0(u) + \sum_{i=1}^n \kappa_i x_i$, where κ_i are arbitrary constants and*

$$f_0''(u) = -\rho_{uu} - \frac{1}{2} \sum_{i=1}^n \kappa_i \partial_i H(u, x_1, \dots, x_n).$$

Proof. Let f be a function on \mathbb{R}^{n+2} . Then the gradient is given by $\nabla f = (\partial_v f, \partial_u f - H \partial_v f, \partial_1 f, \dots, \partial_n f)$ and thus (1) becomes

$$(14) \quad \begin{cases} \frac{1}{2} \sum_{i=1}^n \partial_i H \partial_i f + \partial_{uu}^2 f - \frac{1}{2} \partial_u H \partial_v f + \rho_{uu} = \lambda H, \\ \partial_{ui}^2 f - \frac{1}{2} \partial_i H \partial_v f = 0, & 1 \leq i \leq n, \\ \partial_{ii}^2 f = \lambda, & 1 \leq i \leq n, \\ \partial_{uv}^2 f = \lambda, \\ \partial_{ij}^2 f = \partial_{vi}^2 f = \partial_{vj}^2 f = 0, & 1 \leq i \neq j \leq n. \end{cases}$$

Integrating equations $\partial_{vi}^2 f = \partial_{vj}^2 f = 0$ in (14) we obtain that the potential function splits as $f(u, v, x_1, \dots, x_n) = f_0(u, x_1, \dots, x_n) + v f_1(u)$ for some functions f_0, f_1 . Moreover equations $\partial_{uv}^2 f = \lambda$ and $\partial_{ij}^2 f = 0$ now show that $f(u, v, x_1, \dots, x_n) = \sum_{i=1}^n f_i(u, x_i) + v(\lambda u + \kappa)$ for some constant κ and functions $f_i, i = 1, \dots, n$. Hence (14) reduces to

$$(15) \quad \begin{cases} \frac{1}{2} \sum_{i=1}^n \partial_i H \partial_i f_i + \sum_{i=1}^n \partial_{uu}^2 f_i - \frac{1}{2} (\lambda u + \kappa) \partial_u H + \rho_{uu} = \lambda H, \\ \partial_{ui}^2 f_i - \frac{1}{2} (\lambda u + \kappa) \partial_i H = 0, & 1 \leq i \leq n, \\ \partial_{ii}^2 f_i = \lambda, & 1 \leq i \leq n. \end{cases}$$

Integrating the last equations in (15) we have

$$f_i(u, x_i) = f_{0,i}(u) + x_i \kappa_i(u) + \frac{\lambda}{2} x_i^2,$$

for some functions $f_{0,i}(u)$ and $\kappa_i(u)$. Substituting the above into (15) and differentiating the second set of equations one gets

$$0 = \partial_{iii}^3 f_i = (\lambda u + \kappa) \partial_{ii}^2 H,$$

which shows that either, $\partial_{ii}^2 H = 0$ for all i (and hence the pp -wave is Ricci flat) or otherwise $\lambda = \kappa = 0$.

The first case, when (M, g_{ppw}) is Ricci flat, was already analyzed in Theorem 6. The second case, $\lambda = \kappa = 0$ shows that non Einstein gradient Ricci solitons are steady and f becomes $f(u, v, x_1, \dots, x_n) = f_0(u) + \sum_{i=1}^n \kappa_i(u) x_i$. Now the second equation in (15) reduces to $\kappa_i'(u) = 0$ and hence $\kappa_i(u) = \kappa_i$ for real constants κ_i , which gives

$$(16) \quad f(u, v, x_1, \dots, x_n) = f_0(u) + \sum_{i=1}^n \kappa_i x_i.$$

Finally, it follows from the first equations in (15) that the function $f_0(u)$ is given by the differential equation

$$(17) \quad f_0'' = \frac{1}{2} \left(\sum_i \partial_{ii}^2 H \right) - \frac{1}{2} \sum_i \kappa_i \partial_i H = -\rho_{uu} - \frac{1}{2} \sum_i \kappa_i \partial_i H,$$

which completes the proof. \square

Remark 13. In general, equation (17) does not have a solution, since the derivatives $\partial_i H(u, x_1, \dots, x_n)$ and $\partial_{ii} H$ may be functions of the x_i 's. Further note that ∇f is not isotropic in general since $\|\nabla f\| = \sum_{i=1}^n \kappa_i^2$, although it is a geodesic vector field since $\nabla_{\nabla f} \nabla f = -\text{Ric}(\nabla f) = 0$.

Remark 14. A special class of pp -waves are *plane waves*, which are defined by setting

$$(18) \quad H(u, x_1, \dots, x_n) = \sum_{i,j} a_{ij}(u) x_i x_j.$$

Note that any plane wave is a isotropic steady gradient Ricci soliton for a potential function f given by (16) for constants $\kappa_i = 0$ for all $i = 1, \dots, n$, since (17) becomes

$$f_0''(u) = \frac{1}{2} \left(\sum_i a_{ii}(u) \right).$$

Moreover, it is shown in [8] that plane waves are geodesically complete and therefore since ∇f is a geodesic vector field, it follows that ∇f is complete.

5.1. Locally conformally flat pp -waves. It follows from the expressions (12) and (13), that a pp -wave is locally conformally flat if and only if the defining function H takes the form

$$(19) \quad H(u, x_1, \dots, x_n) = a(u) \sum_{i=1}^n x_i^2 + \sum_{i=1}^n b_i(u) x_i + c(u),$$

where a, b_1, \dots, b_n, c are smooth functions of one variable.

In this case condition (17) reduces to

$$(20) \quad f_0'' = -\rho_{uu} - \frac{1}{2} \sum_{i=1}^n \kappa_i b_i(u) - a(u) \sum_{i=1}^n \kappa_i x_i,$$

where $\rho_{uu} = -na(u)$. So, if we differentiate (20) with respect to x_i we get that $a(u)\kappa_i = 0$ for all $i = 1, \dots, n$. Then, unless the manifold is flat, it follows that necessarily $\kappa_i = 0$ for all i , and the potential function is given by

$$f(u, v, x_1, \dots, x_n) = f_0(u), \quad \text{where} \quad f_0''(u) = -\rho_{uu} = na(u).$$

This completes the proof of Theorem 1.

Remark 15. Gradient Ricci solitons are a particular family of Ricci solitons, which are triples (M, g, X) where X is a vector field satisfying:

$$(21) \quad \frac{1}{2} \mathcal{L}_X g + \rho = \lambda g.$$

Here \mathcal{L}_X denotes the Lie derivative in the direction of X . Note that when $X = \nabla f$ equation (21) becomes equation (1).

Let $X = X_u \partial_u + X_v \partial_v + \sum_i X_i \partial_i$ be an arbitrary vector field on $(\mathbb{R}^{n+2}, g_{ppw})$.

Then (21) becomes

$$(22) \quad \left\{ \begin{array}{ll} \frac{1}{2} \sum_{i=1}^n \partial_i H X_i + \frac{1}{2} \partial_u H X_u + H \partial_u X_u + \partial_u X_v + \rho_{uu} = \lambda H, & \\ H \partial_v X_u + \partial_v X_v + \partial_u X_u = 2\lambda, & \\ H \partial_i X_u + \partial_i X_v + \partial_u X_i = 0, & 1 \leq i \leq n, \\ \partial_i X_j + \partial_j X_i = 0, & 1 \leq i \neq j \leq n, \\ \partial_i X_u + \partial_v X_i = 0, & 1 \leq i \leq n, \\ \partial_v X_u = 0; \quad \partial_i X_i = \lambda, & 1 \leq i \leq n. \end{array} \right.$$

Consider the vector field

$$(23) \quad X = \left(p(u) - \sum_{i=1}^n q_i'(u) x_i + 2\lambda v \right) \partial_v + \sum_{i=1}^n (q_i(u) + \lambda x_i) \partial_i,$$

where functions p and q_i satisfy the following conditions

$$(24) \quad \left\{ \begin{array}{ll} a(u) q_i(u) - q_i''(u) = \frac{\lambda}{2} b_i(u), & 1 \leq i \leq n, \\ \frac{1}{2} \sum_{i=1}^n b_i(u) q_i(u) + \rho_{uu} + p'(u) = \lambda c(u). \end{array} \right.$$

Note that one can always find p, q_i being solutions of (24). A straightforward calculation from (22) shows that (M, g, X) is a Ricci soliton. Also observe that λ is the constant of equation (21) and can be chosen with absolute freedom. In contrast to the gradient case, we obtain that *any locally conformally flat pp-wave (M, g) admits appropriate vector fields resulting in expanding, steady and shrinking Ricci solitons.*

5.2. Lorentzian manifolds with recurrent curvature. A pseudo-Riemannian manifold (M, g) is said to be *recurrent* (or with *recurrent curvature*) if $\nabla R = \sigma \otimes R$ for some 1-form σ .

Observe here that *locally conformally flat pp-waves are recurrent.* Indeed it suffices to show that the Ricci tensor is recurrent due to local conformal flatness. For any locally conformally flat pp-wave (M, g_{ppw}) as in (19) the only nonzero component of the Ricci tensor is $\rho_{uu} = -na(u)$, and hence from (11) it follows that the only nonzero component of $\nabla \rho$ becomes $\nabla_{\partial_u} \rho_{uu} = -na'(u)$. Hence $\nabla \rho = \sigma \otimes \rho$, just considering the 1-form $\sigma = (\ln a(u))' du$.

Recurrent Lorentz manifolds have been classified by Walker [27] (see also Galaev [16]). Non-symmetric Lorentzian recurrent manifolds are pp-waves which correspond to one of the following two families

Type I: The defining function satisfies $H(u, x_1, \dots, x_n) = H(u, x_1)$ where $\partial_{11}^2 H$ is not constant.

Type II: The defining function is given by $H(u, x_1, \dots, x_n) = a(u) (\sum_{i=1}^n b_i x_i^2)$ for constants b_1, \dots, b_n with $|b_1| \geq \dots \geq |b_n|$, $b_2 \neq 0$, and a function h such that $a'(u) \neq 0$.

For a recurrent manifold of Type I, condition (17) reduces to

$$(25) \quad f_0''(u) = -\rho_{uu} - \frac{1}{2} \kappa_1 \partial_1 H(u, x_1),$$

where $\rho_{uu} = -\frac{1}{2} \partial_{11}^2 H(u, x_1)$.

Differentiating in (25) with respect to x_1 we get $\frac{\kappa_1}{2} \partial_{11}^2 H(u, x_1) - \frac{1}{2} \partial_{111}^3 H(u, x_1) = 0$, and hence the defining function $H(u, x_1)$ becomes

$$H(u, x_1) = \frac{1}{\kappa_1^2} e^{\kappa_1 x_1} h_0(u) + h_1(u) + x_1 h_2(u).$$

Then the soliton is given by (see (16)-(17)) $f(u, v, x_1, \dots, x_n) = f_0(u) + \sum_{i=1}^n \kappa_i x_i$, where

$$f_0''(u) = -\frac{\kappa_1}{2} h_2(u).$$

Note that in this case ∇f is always spacelike (since $\kappa_1 \neq 0$) and the underlying manifold is not locally conformally flat (unless it is flat which occurs if $h_0(u) = 0$).

For a recurrent manifold of Type II condition (17) reduces to

$$(26) \quad f_0''(u) = -\rho_{uu} - a(u) \sum_{i=1}^n \kappa_i b_i x_i$$

where $\rho_{uu} = -a(u) \sum_i b_i$. Taking derivatives with respect to x_i in (26) we get that $\kappa_i b_i a(u) = 0$ for all i and therefore, as $b_1 \neq 0 \neq b_2$ there are two different possibilities:

- (a) If $b_i \neq 0$ for all i then, unless the manifold is flat, it follows that $\kappa_i = 0$ for all i and the potential function is given by $f(u, v, x_1, \dots, x_n) = f_0(u)$ where $f_0''(u) = -\rho_{uu} = a(u) \sum_{i=1}^n b_i$. In this case the Ricci soliton is isotropic ($\|\nabla f\| = 0$).
- (b) If $b_j = 0$ for some $j \in \{3, \dots, n\}$, then $\kappa_i = 0$ for $i < j$ and the potential function is given by

$$f(u, v, x_1, \dots, x_n) = f_0(u) + \sum_{i=j}^n \kappa_i x_i, \text{ where } f_0''(u) = -\rho_{uu} = a(u) \sum_{i=1}^j b_i.$$

Further observe that in this case ∇f is spacelike.

Summarizing the above, we have that

Type I: A recurrent Lorentzian manifold of Type I admits a function f resulting in a gradient Ricci soliton if and only if the defining function $H(u, x_1)$ satisfies $H(u, x_1) = \frac{1}{\kappa_1^2} e^{\kappa_1 x_1} h_0(u) + h_1(u) + x_1 h_2(u)$. Moreover, in this case ∇f is a spacelike vector field.

Type II: A recurrent Lorentzian manifold of Type II always admits a function f resulting in a gradient Ricci soliton. The causal character of ∇f may be null or spacelike.

5.3. Two-symmetric Lorentzian manifolds. As a generalization of locally symmetric spaces, Lorentzian manifolds whose higher order derivatives of the curvature tensor vanish, have been investigated. A Lorentzian manifold is said to be *two-symmetric* if $\nabla^2 R = 0$ but $\nabla R \neq 0$. It was shown by Senovilla [26] that two-symmetric Lorentzian manifolds admit a parallel null vector field and the local structure of such manifolds was given recently in [3], [1] showing that they are a special family of *pp*-waves.

An $(n+2)$ -dimensional Lorentzian manifold is two-symmetric if and only if it is a *pp*-wave as in (9) with

$$(27) \quad H(u, x_1, \dots, x_n) = \sum_{i,j=1}^n (a_{ij}u + b_{ij})x_i x_j,$$

where (a_{ij}) is a diagonal matrix with the diagonal elements $a_{11} \leq \dots \leq a_{nn}$ non-zero real numbers and (b_{ij}) an arbitrary symmetric matrix of real numbers.

Now, an immediate application of (16)-(17) shows that two-symmetric Lorentzian manifolds are gradient Ricci solitons whose potential function is given by $f = f_0(u)$ where

$$f_0''(u) = -\rho_{uu} = \sum_{i=1}^n (b_{ii} + u a_{ii}).$$

Finally observe that ∇f is a geodesic vector field and (M, g) is geodesically complete. Moreover ∇f is isotropic.

5.4. Conformally symmetric Lorentzian manifolds. A Lorentzian manifold is said to be *conformally symmetric* if the covariant derivative of the Weyl tensor vanishes identically ($\nabla W = 0$). Clearly locally symmetric and locally conformally flat manifolds are conformally symmetric and the converse is true in the Riemannian setting. In Lorentzian signature there exist, however, conformally symmetric manifolds which are neither locally conformally flat nor locally symmetric. These

manifolds have recurrent Ricci tensor and have been described locally by Derdzinski and Roter [13]. It turns out that all of them are pp -waves given by

$$H(u, x_1, \dots, x_n) = a(u) \sum_i x_i^2 + \sum_{i,j} b_{ij} x_i x_j$$

where (b_{ij}) is a nonzero symmetric matrix with $\sum_{i=1}^n b_{ii} = 0$.

Now equations (16) and (17) show that any conformally symmetric Lorentzian manifold of this family admits a function f resulting in a steady gradient Ricci soliton with isotropic ∇f .

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